Math 601 – **Spring** 2014
Homework #3

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1. (Hatcher, number 2.1.14) Determine whether there exists a short exact sequence

$$0 \to \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \to 0.$$

More generally, determine which abelian groups A fit into a short exact sequence

$$0 \to \mathbb{Z}_{p^m} \to A \to \mathbb{Z}_{p^n} \to 0$$

with p prime. What about the case of short exact sequences

$$0 \to \mathbb{Z} \to A \to \mathbb{Z}_n \to 0?$$

2. (Hatcher, number 2.1.20) Show that $\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(SX)$ for all n, where SX is the suspension of X; $SX = X \times I/(x,1) \sim (y,1)$ and $(x,0) \sim (y,0)$ for all $x, y \in X$.

3. (Hatcher, number 2.1.26) Show that $H_1(X, A)$ is not isomorphic to $\widetilde{H}_1(X/A)$ if $X = [0,1]$ and A is the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$ together with its limit 0.

4. (Hatcher, number 2.2.2) Given a map $f : S^{2n} \longrightarrow S^{2n}$, show that there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbb{RP}^{2n} \longrightarrow \mathbb{RP}^{2n}$ has a fixed point. Construct maps $\mathbb{RP}^{2n-1} \longrightarrow \mathbb{RP}^{2n-1}$ without fixed points from linear transformations $R^{2n} \longrightarrow \mathbb{R}^{2n}$ without eigenvectors.

5. (Hatcher, number 2.2.8) A polynomial $f(z)$ with complex coefficients, viewed as a map $\mathbb{C} \longrightarrow \mathbb{C}$, can always be extended to a continuous map of one-point compactifications $\hat{f} : S^2 \longrightarrow S^2$. Show that the degree of $\hat{f}$ equals the degree of f as a polynomial. Show also that the local degree of $\hat{f}$ at a root of f is the multiplicity of the root.

6. Prove the Brouwer fixed point theorem: Any continuous map $f : \mathbb{D}^n \longrightarrow \mathbb{D}^n$ admits a fixed point.

7. By considering T^2 as $\mathbb{R}^2/\mathbb{Z}^2$, we see that $SL(2,\mathbb{Z})$ acts on T^2 via homeomorphisms. For $\phi \in SL(2,\mathbb{Z})$, let $Y_\phi = (S^1 \times \mathbb{D}^2) \cup_\phi (S^1 \times \mathbb{D}^2)$. Determine $H_*(Y_\phi)$ in terms of ϕ.

8. Given $\phi \in SL(2,\mathbb{Z})$, consider the mapping torus of T^2 with respect to ϕ:
$X_\phi = T^2 \times I/(x,1) \sim (\phi(x),0)$. Compute $H_*(X_\phi)$ in terms of ϕ.

9. For any knot in $\mathbb{R}^3$, compute $H_*(\mathbb{R}^3 \backslash \nu K)$, where νK is a neighborhood of the knot.